Name: ______________________________

Teacher: ______________________________

Class: ______________________________

FORT STREET HIGH SCHOOL

2013

HIGHER SCHOOL CERTIFICATE COURSE
ASSESSMENT TASK 2

Mathematics Extension 1

TIME ALLOWED: 1½ HOURS
(PLUS 5 MINUTES READING TIME)

Outcomes Assessed	Questions	Marks
Determines integrals by reduction to a standard form through a given substitution.	1, 2, 3	
Manipulates algebraic expressions to solve problems involving inverse functions	4, 5, 6	
Synthesises mathematical solutions to harder problems and communicates them in appropriate form	7, 8	

Question	1	2	3	4	5	6	7	8	Total	%
Marks	/8	/8	/9	/8	/9	/8	/8	/7	/65	

Directions to candidates:

- Attempt all questions
- The marks allocated for each question are indicated
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Board – approved calculators may be used
- Each new question is to be started in a new booklet

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Question 1 *8 marks*

a. Sketch, on the same diagram, the curves $y=\sin x$ and $y=\sin^2 x$ for $0 \le x \le \pi$ — 2 marks

b. Find the area bounded by the curves in the domain. — 2 marks

c. Show that $\sin^2 x-\sin^4 x=\frac{1}{4}\sin^2 2x$ — 2 marks

d. Find the volume generated when the area from part b is rotated about the x axis. — 2 marks

Question 2 *8 marks*

a.

$\int x\sqrt{3-x}\,dx$ using the substitution $u=3-x$ — 3 marks

b.

Without the use of a calculator, give the exact value of : $\tan\left\{\sin^{-1}\left(\frac{12}{13}\right)\right\}$ — 2 marks

(Show working)

c.

Find the inverse function of $f(x)=(x-1)^3$ and sketch both curves. — 3 marks

Question 3 *9 marks*

a.

Evaluate

$$\int_0^{\frac{\pi}{2}} \cos^2 4x \; dx$$

4 marks

b.

Show that

$$\int_0^{\frac{1}{4}} \frac{3x}{(1+4x)^3}\, dx=\frac{3}{128},$$ using the substitution $u=1+4x$

5 marks

Question 4 *8 marks*

a.

Differentiate the following with respect to x

i. $y = \sin^{-1}\left(x^2\right)$ — 2 marks

ii. $y = \ln\left(\tan^{-1} x\right)$ — 2 marks

b. — 2 marks

Show that if $y = e^{\operatorname{cosec} x}$, then $\dfrac{dy}{dx} = -\cot x \operatorname{cosec} x \, e^{\operatorname{cosec} x}$

c.

Find the equation of the normal to the curve $y = \tan^{-1} x$ at the point where $x = 1$. — 2 marks

Question 5 *9 marks*

a.

Find the primitive functions of

i. $\dfrac{x}{4+x^2}$ — 2 marks

ii. $\dfrac{-1}{\sqrt{1-4x^2}}$ — 2 marks

b.

i. Show that $\dfrac{2x^2+5}{\left(4+x^2\right)\left(1+x^2\right)} = \dfrac{1}{4+x^2} + \dfrac{1}{1+x^2}$ — 1 mark

ii. Hence evaluate $\displaystyle\int_0^1 \frac{2x^2+5}{\left(4+x^2\right)\left(1+x^2\right)}dx$ correct to 2 decimal places — 2 marks

c.

The area bounded by the curve $y = \dfrac{1}{\sqrt{4+x^2}}$, the x axis and the lines $x = -2$ and $x = 2$ is rotated about the x axis. Find the volume of the solid generated. — 2 marks

Question 6 *8 marks*

a.

i. Find the domain and range of $y = 5\sin^{-1}\frac{x}{3}$ 2 marks

ii. Sketch the curve 2 marks

b.

Calculate leaving your answer in exact form 4 marks

$$\int_{\frac{1}{8}}^{\frac{\sqrt{3}}{8}} \frac{dx}{\sqrt{1-16x^2}}$$

Question 7 *8 marks*

a.

i. If $y = \tan^{-1} x$, express x as a function of y 1 mark

ii. Prove that 4 marks

$$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$$

b.

Find the function whose differential is 3 marks

$$\frac{x^2+12}{x^2+9}$$

Question 8 *7marks*

a.

On the same set of axes sketch 2 marks

$f(x) = \sin^{-1} x$ and $g(x) = \cos^{-1} x$.

b.

If $h(x) = \sin^{-1} x \cos^{-1} x$, find $h'(x)$. 1 mark

c.

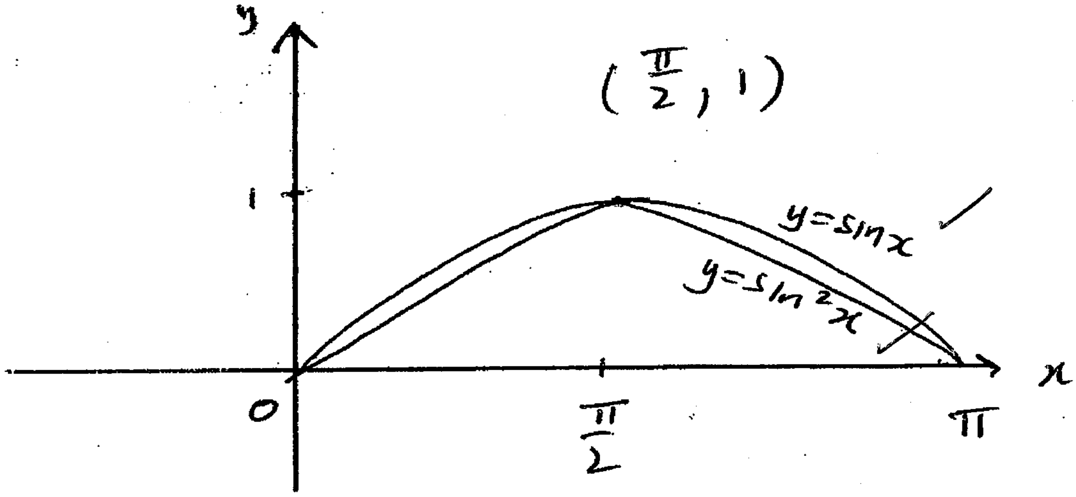
Sketch $h(x)$. Find and label the stationary point. 4 marks

Assessment Task 2 2013

Extension 1

Question 1

a.



b.

$$Area = \int_0^{\pi} \sin x - \sin^2 x \; dx$$

$$= -[\cos x]_0^{\pi} - \frac{1}{2}\int_0^{\pi} 1 - \cos 2x \; dx$$

$$= -[\cos x]_0^{\pi} - \frac{1}{2}\left[x - \frac{1}{2}\sin 2x\right]_0^{\pi}$$

$$= -(-1-1) - \frac{1}{2}\left[(\pi - 0) - (0-0)\right]$$

$$= 2 - \frac{\pi}{2} units^2$$

c.

$$LHS = \sin^2 x - \sin^4 x$$

$$= \sin^2 x\left(1 - \sin^2 x\right)$$

$$= \sin^2 x \times \cos^2 x$$

$$= (\sin x \cos x)^2$$

$$= \left(\frac{1}{2}\sin 2x\right)^2$$

$$= \frac{1}{4}\sin^2 2x$$

d.

$$V = \pi \int_0^{\pi} \sin^2 x - \sin^4 x \; dx$$

$$= \pi \int_0^{\pi} \frac{1}{4}\sin^2 2x \; dx$$

$$= \frac{\pi}{4}\left[\frac{1}{2}(1 - \cos 4x)\right]_0^{\pi}$$

$$= \frac{\pi}{8}\left[x - \frac{1}{4}\sin 4x\right]_0^{\pi}$$

$$= \frac{\pi}{8}\left[\pi - 0 - (0-0)\right]$$

$$= \frac{\pi^2}{8} units^3$$

Question 2

a.

$$u=3-x$$
$$du=-dx$$
$$dx=-du$$

$$I=\int x\sqrt{3-x}\ dx$$

$$I=\int(3-u)\times\sqrt{u}\times -du$$
$$=-\int(3-u)u^{\frac{1}{2}}\ du$$
$$=\int(u-3)u^{\frac{1}{2}}du$$
$$=\int u^{\frac{3}{2}}-3u^{\frac{1}{2}}du$$
$$=\frac{2u^{\frac{5}{2}}}{5}-2u^{\frac{3}{2}}+C$$
$$=\frac{2}{5}(3-x)^{\frac{5}{2}}-2(3-x)^{\frac{3}{2}}+C$$

b.

Let

$$\alpha=\sin^{-1}\left(\frac{12}{13}\right)$$

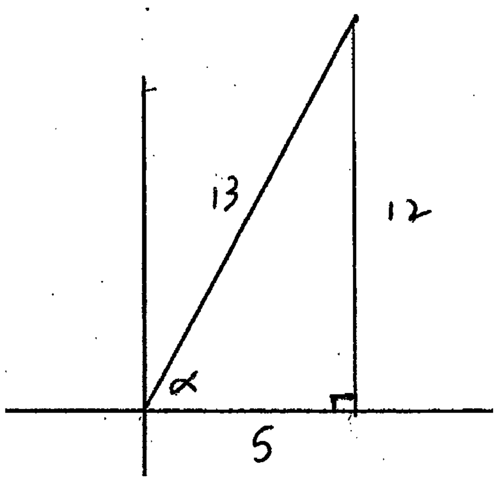
$$\sin\alpha=\frac{12}{13}$$

from the diagram

$$\tan\alpha=\frac{12}{5}$$

since $\alpha=\sin^{-1}\left(\frac{12}{13}\right)$

then $\tan\{\sin^{-1}\left(\frac{12}{13}\right)\}=\frac{12}{5}$

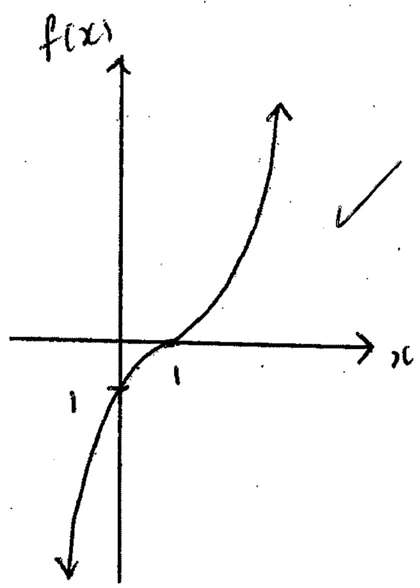


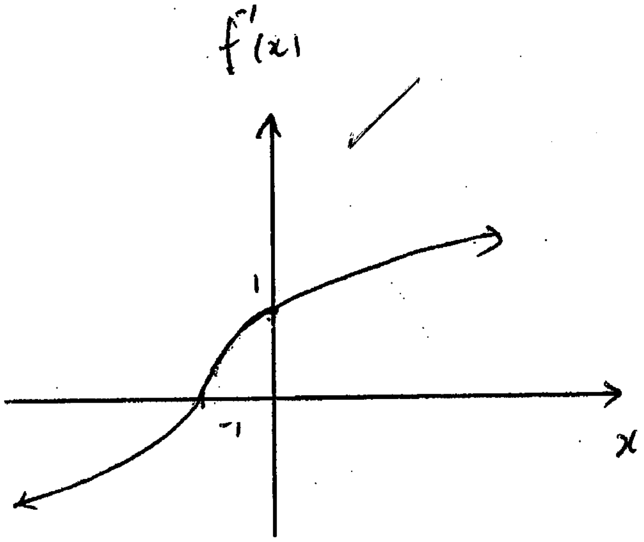
c.

$f(x): y = (x-1)^3$

$f^{-1}(x): \quad x = (y-1)^3$

$y = x^{\frac{1}{3}} + 1$





Comments :

Question 3

a.

$$\int_0^{\frac{\pi}{2}} \cos^2 4x \, dx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos 8x + 1 \, dx$$

$$= \frac{1}{2}\left[\frac{1}{8}\sin 8x + x\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left[\frac{1}{8}\sin 4\pi + \frac{\pi}{2} - (0-0)\right]$$

$$= \frac{\pi}{4}$$

$$\cos 8x = \cos^2 4x - \sin^2 4x$$

$$= \cos^2 4x - \left(1 - \cos^2 x\right)$$

$$= 2\cos^2 4x - 1$$

$$\frac{1}{2}(\cos 8x + 1) = \cos^2 4x$$

b.

Let

$$I = \int_0^{\frac{1}{4}} \frac{3x}{(1+4x)^3} dx$$

$$= \frac{3}{16}\int_1^2 \frac{3(u-1)}{4} \times \frac{1}{u^3} \times \frac{du}{4}$$

$$= \frac{3}{16}\int_1^2 \frac{u-1}{u^3} du$$

$$= \frac{3}{16}\int_1^2 u^{-2} - u^{-3} du$$

$$= \frac{3}{16}\left[-u^{-1} + \frac{u^{-2}}{2}\right]_1^2$$

$$= \frac{3}{16}\left[\frac{1}{2u^2} - \frac{1}{u}\right]_1^2$$

$$= \frac{3}{16}\left[-\frac{1}{2} + \frac{1}{8} - \left(-1 + \frac{1}{2}\right)\right]$$

$$= \frac{3}{128}$$

$$u = 1 + 4x \qquad u - 1 = 4x$$

$$\frac{du}{dx} = 4 \qquad 3\left(\frac{u-1}{4}\right) = 3x$$

$$dx = \frac{du}{4}$$

$$x = \frac{1}{4} \quad u = 2$$

$$x = 0 \quad u = 1$$

Question 4

a.

i.

$$y = \sin^{-1}\left(x^2\right)$$

$$y' = \frac{1}{\sqrt{1-\left(x^2\right)^2}} \times 2x$$

$$y' = \frac{2x}{\sqrt{1-x^4}}$$

ii.

$$y = \ln\left(\tan^{-1} x\right)$$

$$y' = \frac{1}{1+x^2} \div \tan^{-1} x$$

$$y' = \frac{1}{\left(1+x^2\right)\tan^{-1} x}$$

b.

$$y = e^{\operatorname{cosec} x}$$

$$y = e^u \qquad \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{dx} = e^u \times \operatorname{cosec} x \cot x$$

$$\frac{dy}{dx} = \operatorname{cosec} x \cot x \, e^{\operatorname{cosec} x}$$

Let

$$u = \operatorname{cosec} x$$

$$u = \frac{1}{\sin x}$$

$$\frac{du}{dx} = \frac{\cos x}{\sin^2 x}$$

$$\frac{du}{dx} = \operatorname{cosec} x \cot x$$

c.

When

$x = 1$

$$y = \tan^{-1}(1) = \frac{\pi}{4}$$

$$\frac{dy}{dx} = \frac{1}{1+x^2}$$

$$f'(1) = \frac{1}{2}$$

the normal has gradient

$m = -2$

$$y - \frac{\pi}{4} = -2(x-1)$$

$$y - \frac{\pi}{4} = -2x + 2$$

$$y = -2x + 2 - \frac{\pi}{4}$$

Question 5

a.

i.

$$\int \frac{x}{4+x^2}\,dx$$
$$= \frac{1}{2}\int \frac{2x}{4+x^2}\,dx$$
$$= \frac{1}{2}\ln\left(4+x^2\right)+C$$

ii.

$$\int \frac{-1}{\sqrt{1-4x^2}}\,dx$$
$$= \frac{1}{2}\int \frac{-1}{\sqrt{\frac{1}{4}-x^2}}\,dx$$
$$= \frac{1}{2}\int \frac{-1}{\sqrt{\left(\frac{1}{2}\right)^2-x^2}}\,dx$$
$$= \frac{1}{2}\cos^{-1}(2x)+C$$

b.

i.

$$RHS = \frac{1}{4+x^2}+\frac{1}{1+x^2}$$
$$= \frac{1+x^2+4+x^2}{\left(4+x^2\right)\left(1+x^2\right)}$$
$$= \frac{2x^2+5}{\left(4+x^2\right)\left(1+x^2\right)}$$
$$= LHS$$

ii.

$$\int_0^1 \frac{2x^2+5}{\left(4+x^2\right)\left(1+x^2\right)}\,dx$$
$$= \int_0^1 \frac{1}{4+x^2}+\frac{1}{1+x^2}\,dx$$
$$= \left[\tan^{-1}\frac{x}{2}+\tan^{-1}x\right]_0^1$$
$$= \tan^{-1}\frac{1}{2}+\tan^{-1}1-\left(\tan^{-1}0+\tan^{-1}0\right)$$
$$= 1.25$$

c.

$$V = \pi\int_{-1}^{2}\frac{1}{4+x^2}\,dx$$
$$= 2\pi\left[\tan^{-1}\left(\frac{x}{2}\right)\right]_0^2$$
$$= 2\pi\left[\tan^{-1}1-\tan^{-1}0\right]$$
$$= 2\pi\times\frac{\pi}{4}$$
$$= \frac{\pi^2}{2}\,units^3$$

Question 6

a.

i. Domain

$$-1 \le \frac{x}{3} \le 1$$

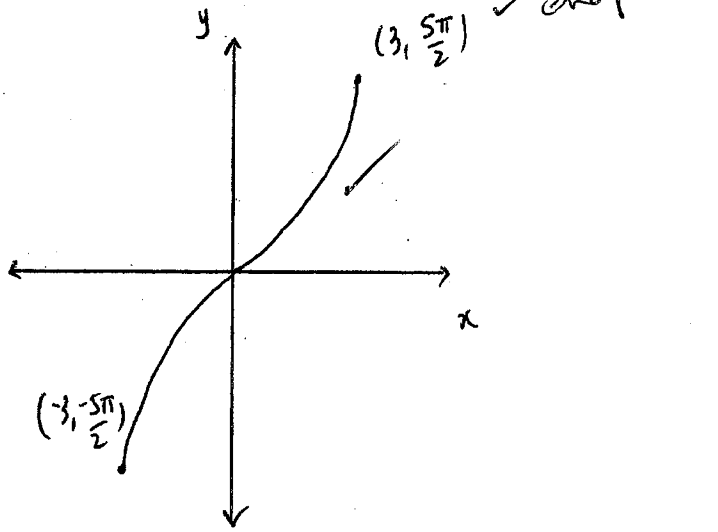
$$-3 \le x \le 3$$

Range

$$-\frac{\pi}{2} \le \frac{y}{5} \le \frac{\pi}{2}$$

$$-\frac{5\pi}{2} \le y \le \frac{5\pi}{2}$$

ii.



b.

$$\int_{\frac{1}{8}}^{\frac{\sqrt{3}}{8}} \frac{dx}{\sqrt{1-16x^2}}$$

$$= \frac{1}{4}\int_{\frac{1}{8}}^{\frac{\sqrt{3}}{8}} \frac{dx}{\sqrt{\left(\frac{1}{4}\right)^2 - x^2}}$$

$$= \frac{1}{4}\left[\sin^{-1}(4x)\right]_{\frac{1}{8}}^{\frac{\sqrt{3}}{8}}$$

$$= \frac{1}{4}\left[\sin^{-1}\frac{\sqrt{3}}{2} - \sin^{-1}\frac{1}{2}\right]$$

$$= \frac{1}{4}\left(\frac{\pi}{3} - \frac{\pi}{6}\right)$$

$$= \frac{\pi}{24}$$

Comments

Question 7

a.

$$y = \tan^{-1} x$$
$$x = \tan y$$

b.

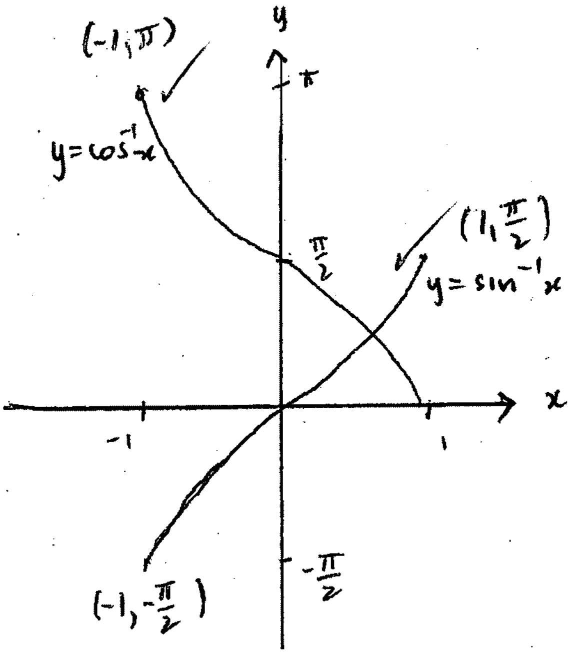
$$y = \tan^{-1} x$$
$$x = \tan y$$
$$\frac{dx}{dy} = \sec^2 y$$
$$\frac{dx}{dy} = 1 + \tan^2 y$$
$$\frac{dx}{dy} = 1 + x^2$$
$$\frac{dy}{dx} = \frac{1}{1+x^2}$$

c.

$$\int \frac{x^2+12}{x^2+9} dx$$
$$= \int \frac{x^2+9+3}{x^2+9} dx$$
$$= \int 1 + \frac{3}{x^2+9} \, dx$$
$$= x + 3\tan^{-1}\frac{x}{3} + C$$

Question 8

a.



b.

$$h(x) = \sin^{-1} x \ \cos^{-1} x$$

$$u(x) = \sin^{-1} x \quad v(x) = \cos^{-1} x$$

$$u'(x) = \frac{1}{\sqrt{1-x^2}} \quad v'(x) = \frac{-1}{\sqrt{1-x^2}}$$

$$h'(x) = \sin^{-1} x \times \frac{-1}{\sqrt{1-x^2}} + \cos^{-1} x \times \frac{1}{\sqrt{1-x^2}}$$

$$h'(x) = \frac{\cos^{-1} x - \sin^{-1} x}{\sqrt{1-x^2}}$$

c.

stationary point occurs when

$h'(x) = 0$

$$0 = \frac{\cos^{-1} x - \sin^{-1} x}{\sqrt{1-x^2}}$$

this can only happen if

$0 = \cos^{-1} x - \sin^{-1} x$

$\cos^{-1} x = \sin^{-1} x$

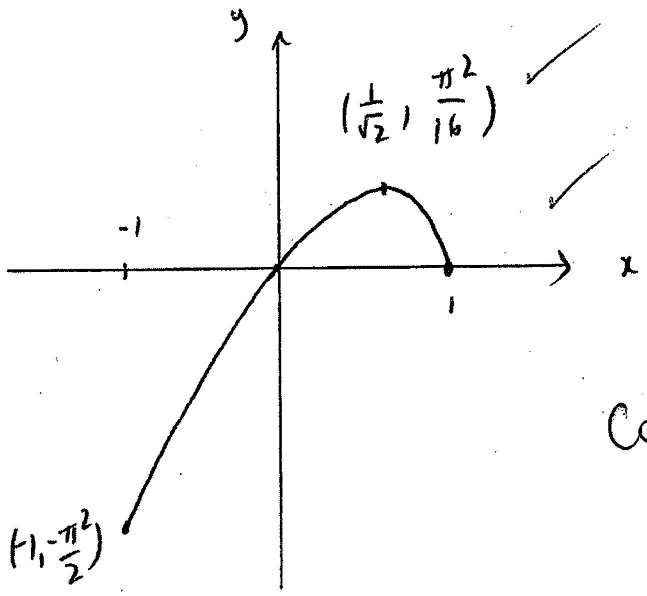
$$x = \frac{1}{\sqrt{2}}$$

when $\cos x = \sin x$

$$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} \qquad \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

then $\frac{\pi}{4} = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) \quad \frac{\pi}{4} = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$

$$\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$$



Comments :